# NUMERICAL ANALYSIS

## Formula Sheet

$$b_n - a_n = \frac{b_0 - a_0}{2^n}$$

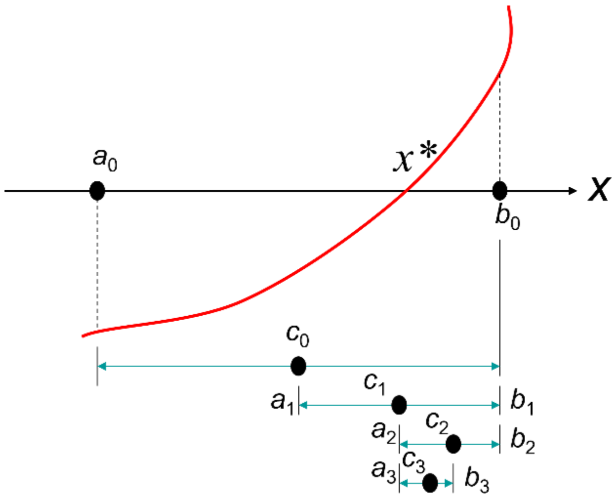
$$\left|c_n - x^*\right| \le c_n - a_n$$

$$\le \frac{b_0 - a_0}{2^{n+1}}$$

For the previous example

$$\frac{0.4}{2^{n+1}} < 10^{-4}$$

$$\boxed{n = 11}$$




Number of Iterations

$$\left|c_n - x^*\right| \le c_n - a_n \qquad \le \frac{b_0 - a_0}{2^{n+1}}$$

Newton Raphson Method

$$x_n = x_{n-1} - \frac{f\left(x_{n-1}\right)}{f'\left(x_{n-1}\right)}$$

Secant Method

$$x_n = \frac{x_{n-1} f\left(x_{n-2}\right) - x_{n-2} f\left(x_{n-1}\right)}{f\left(x_{n-2}\right) - f\left(x_{n-1}\right)}$$

False Position

$$x_n = \frac{x_{n-1} f(x_{n-2}) - x_{n-2} f(x_{n-1})}{f(x_{n-2}) - f(x_{n-1})}$$

Newton Raphson Method

(multivariable)

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \end{bmatrix} - \underline{J}^{-1}\left(x_1^{(k-1)}, x_2^{(k-1)}\right) \begin{bmatrix} f_1\left(x_1^{(k-1)}, x_2^{(k-1)}\right) \\ f_2\left(x_1^{(k-1)}, x_2^{(k-1)}\right) \end{bmatrix}$$

$$\underline{x}^{(k)} = \underline{x}^{(k-1)} - \underline{J}^{-1}\left(\underline{x}^{(k-1)}\right) \underline{f}\left(\underline{x}^{(k-1)}\right)$$

ITERATIVE METHODS (I): Gauss-Jacobi Method

$$\begin{aligned}
x_1^{(k)} &= \frac{1}{a_{11}}\left(b_1 - a_{12} x_2^{(k-1)} - a_{13} x_3^{(k-1)} - \cdots\cdots - a_{1n} x_n^{(k-1)}\right) \\
x_2^{(k)} &= \frac{1}{a_{22}}\left(b_2 - a_{21} x_1^{(k-1)} - a_{23} x_3^{(k-1)} - \cdots\cdots - a_{2n} x_n^{(k-1)}\right) \\
x_3^{(k)} &= \frac{1}{a_{33}}\left(b_3 - a_{31} x_1^{(k-1)} - a_{32} x_2^{(k-1)} - \cdots\cdots - a_{3n} x_n^{(k-1)}\right) \\
&\vdots \\
x_n^{(k)} &= \frac{1}{a_{nn}}\left(b_n - a_{n1} x_1^{(k-1)} - a_{n2} x_2^{(k-1)} - \cdots\cdots - a_{n,n-1} x_{n-1}^{(k-1)}\right)
\end{aligned}$$

ITERATIVE METHODS (I): Gauss-Seidel Method

$$\begin{aligned}
x_1^{(k)} &= \frac{1}{a_{11}}\left(b_1 - a_{12} x_2^{(k-1)} - a_{13} x_3^{(k-1)} - \cdots\cdots - a_{1n} x_n^{(k-1)}\right) \\
x_2^{(k)} &= \frac{1}{a_{22}}\left(b_2 - a_{21} x_1^{(k)} - a_{23} x_3^{(k-1)} - \cdots\cdots - a_{2n} x_n^{(k-1)}\right) \\
x_3^{(k)} &= \frac{1}{a_{33}}\left(b_3 - a_{31} x_1^{(k)} - a_{32} x_2^{(k)} - \cdots\cdots - a_{3n} x_n^{(k-1)}\right) \\
&\vdots \\
x_n^{(k)} &= \frac{1}{a_{nn}}\left(b_n - a_{n1} x_1^{(k)} - a_{n2} x_2^{(k)} - \cdots\cdots - a_{n,n-1} x_{n-1}^{(k)}\right)
\end{aligned}$$

Convergence of Iterative Methodz; Gauss-Jacobi

$\underline{x}(k) = \underline{D}\text{-}1\underline{B} - \underline{D}\text{-}1(\underline{L}+\underline{U})\ \underline{x}(k\text{-}1)$ $\quad \underline{T} = -\ \underline{D}\text{-}1(\underline{L}+\underline{U})$ $\quad \underline{C} = \underline{D}\text{-}1\underline{B}$

Gauss-Seidel

$\underline{x}^{(k)} = (\underline{L}+\underline{D})^{-1}\underline{B} - (\underline{L}+\underline{D})^{-1}\underline{U}\,\underline{x}^{(k-1)}$ $\underline{T} = -(\underline{L}+\underline{D})-1\underline{U}$ $\underline{C} = (\underline{L}+\underline{D})-1\underline{B}$

$$\sqrt{\rho\left(\underline{T}^T\underline{T}\right)} < 1$$

Optimal Relaxation Factor
Gauss-Jacobi

$$\underline{T} = \underline{I} - \omega\,\underline{D}^{-1}\underline{A}$$

Gauss-Seidel

$$\underline{T} = (\underline{D} + \omega\,\underline{L})^{-1}[(1-\omega)\underline{D} - \omega\underline{U}]$$

$$\omega_{opt} \square \frac{2}{1+\sqrt{1-\left[\rho\left(\underline{T}_j\right)\right]^2}}$$

Gauss-Jacobi Method

$$\underline{x}^{(k)} = \underline{D}^{-1}\underline{B} - \underline{D}^{-1}(\underline{L}+\underline{U})\,\underline{x}^{(k-1)}$$

Gauss-Seidel Method

$$\underline{x}^{(k)} = (\underline{D}+\underline{L})^{-1}\underline{B} - (\underline{D}+\underline{L})^{-1}\underline{U}\,\underline{x}^{(k-1)}$$

Curve-Fitting

$$\begin{bmatrix} \sum_{k=1}^{N} 1 & \sum_{k=1}^{N} x_k & \sum_{k=1}^{N} x_k^2 & \cdots & \sum_{k=1}^{N} x_k^n \\ \sum_{k=1}^{N} x_k & \sum_{k=1}^{N} x_k^2 & \sum_{k=1}^{N} x_k^3 & \cdots & \sum_{k=1}^{N} x_k^{n+1} \\ \sum_{k=1}^{N} x_k^2 & \sum_{k=1}^{N} x_k^3 & \sum_{k=1}^{N} x_k^4 & \cdots & \sum_{k=1}^{N} x_k^{n+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \sum_{k=1}^{N} x_k^n & \sum_{k=1}^{N} x_k^{n+1} & \sum_{k=1}^{N} x_k^{n+2} & \cdots & \sum_{k=1}^{N} x_k^{2n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{N} y_k \\ \sum_{k=1}^{N} y_k x_k \\ \sum_{k=1}^{N} y_k x_k^2 \\ \vdots \\ \sum_{k=1}^{N} y_k x_k^n \end{bmatrix}$$

LAGRANGE APPROXIMATION

$$P(x) = \sum_{i=0}^{n} L_i(x)\, y_i \qquad\qquad L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

INTERPOLATION AND POLYNOMIAL APPROXIMATION

Newton Polynomial

**Special case:**

**When $x_0, x_1, \dots, x_n$ are arranged consecutively with equal spacing $h = x_{n+1} - x_n$]**

| $x$ | $f$ | 1st Divided differences | 2nd Divided differences | 3rd Divided differences | 4th Divided differences |
|---|---|---|---|---|---|
| $x_0$ | $f_0$ | | | | |
| | | $\frac{f_1 - f_0}{h} = \frac{\Delta f_0}{h}$ | | | |
| $x_1$ | $f_1$ | | $\frac{\frac{\Delta f_1}{h} - \frac{\Delta f_0}{h}}{2h} = \frac{\Delta^2 f_0}{2h^2}$ | | |
| | | $\frac{f_2 - f_1}{h} = \frac{\Delta f_1}{h}$ | | $\frac{\frac{\Delta^2 f_1}{2h^2} - \frac{\Delta^2 f_0}{2h^2}}{3h} = \frac{\Delta^3 f_0}{6h^3}$ | $\frac{\Delta^4 f_0}{24h^4}$ $\cdots\cdots$ |
| $x_2$ | $f_2$ | | $\frac{\frac{\Delta f_2}{h} - \frac{\Delta f_1}{h}}{2h} = \frac{\Delta^2 f_1}{2h^2}$ | $\vdots$ | $\vdots$ |
| | | $\frac{f_3 - f_2}{h} = \frac{\Delta f_2}{h}$ | $\vdots$ | | |
| $x_3$ | $f_3$ | | | | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots\cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$